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SPATIALLY PERIODIC SUSPENSIONS OF CONVEX PARTICLES IN LINEAR SHEAR FLOWS.

I. DESCRIPTION AND KINEMATICS

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Abstract—Preparatory to a subsequent dynamical study in Part II, aimed at calculating the rheological properties of geometrically-ordered models of concentrated suspensions, a purely kinematical study is here presented of the motion of a mobile spatially periodic array of identical convex particles, typically spheres, participating in a macroscopically homogeneous linear shear flow to which the suspension as a whole is subjected. The geometrical configuration of such particle-lattice suspensions is shown to evolve temporally in a manner dependent upon the initial lattice configuration and the specific bulk shearing motion to which the suspension is subjected. Under certain circumstances the particle-lattice system is found to reproduce itself periodically in time—or, less stringently—"almost" periodically. Precise circumstances under which this occurs are exhaustively delineated for the entire class of two-dimensional isochoric spatially homogeneous shearing motions, parametrized by a scalar λ expressing the relative amounts of shear and vorticity present in the flow. This investigation is performed for both two- and three-dimensional lattices. (Eventual time averaging of the local, instantaneous, dynamical, interstitial fluid properties of these almost self-reproducing systems in Part II furnishes the rheological properties of the suspension.) Using concepts borrowed from Minkowski's geometry of numbers, calculations are outlined for establishing the maximum volume fraction of suspended particles that is kinematically possible for each shearing motion. This is observed to be always less than would obtain in a comparable *static* system.

1. INTRODUCTION

The purpose of the present series of papers, of which this is the first, is to rigorously derive the rheological properties of a spatially periodic suspension of particles dispersed in a fluid (not necessarily Newtonian) undergoing a linear shear flow. This geometrical arrangement is proposed as a tractable mathematical model of a concentrated suspension.

Concentrated suspensions have been the subject of innumerable studies. Many of the results and interpretations issuing therefrom are summarized in the following books and review articles: Happel & Brenner (1965), Brenner (1970), Jeffrey & Acrivos (1976), Buyevich & Shchelchkova (1978), Herczyński & Pieńkowska (1980) and Mewis (1980). Essentially four theoretical focii exist: (i) the dilute limit, including first-order hydrodynamic interactions (Batchelor 1974); (ii) cell models (Happel 1957); (iii) semiempirical models, the archetype of which is due to Mooney (1951); (iv) statistical models (cf. Herczyński & Pieńkowska 1980). Each such analysis suffers from its own special limitations. Dilute limit approaches are faced after the first- or second-order terms with algebraically intractable calculations. The second and third methods are somewhat more successful when compared with experiment, but the underlying assumptions are often arbitrary. Moreover, because of their *ad hoc* nature, they cannot be rationally improved. The last method is faced with the usual problems of statistical models, namely of providing a rational source for the stochastic, nondeterministic hydrodynamic concepts introduced.

Though numerous experimental studies have been performed, most are of limited interest due to the poorly defined suspension characterization. Happily, exceptions exist. Among these are included the carefully delineated investigations of Krieger (1972), Krieger & Eguiluz (1976) and Hoffman (1972, 1974).

A markedly different theoretical approach is used here to analyze the behavior of concentrated suspensions, at least for circumstances where they may be reasonably well approximated as being spatially periodic in their mode of arrangement. The overwhelming

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advantage of this configuration is that it can be made both rigorous and tractable. Its obvious drawback lies in the perfect order which is thereby imposed upon the suspension. Our approach to the topic embodies an unusual mixture of fluid mechanics, number theory and ergodic theory. In fact, the mathematics which will be employed here is somewhat delicate, in the sense that many of the functions arising in subsequent developments are pathological. Discontinuity, for instance, is the rule rather than the exception. This can, perhaps, be traced to the lack of any stochastic features—such as Brownian motion—that might otherwise act to smooth out these discontinuities.

In this first paper of the series, attention will be restricted to the geometrical description and kinematics of a spatially periodic sheared suspension—a subject which does not appear to have been thoroughly addressed before. We have attempted here to make the developments reasonably self-contained, without, however, burdening the reader with excessive detail. Thus, proofs and technical details will generally be relegated to the references cited.

Section 2 reviews fundamental aspects of lattice theory and introduces the norm on a lattice set. The "intersection" of a straight line with the points comprising an integer lattice is discussed in connection with a series of classical theorems.

Section 3 is devoted to regular arrangements of convex bodies situated at lattice points. This is, properly speaking, the subject of the geometry of numbers, pioneered by H. Minkowski (in order to solve and appropriately generalize some problems of quadratic forms posed by C. Hermite). One of the most important questions here is the explicit calculation (or bounding) of the maximum density obtainable by a regular arrangement of rigid convex bodies. In response to the kinematical issues posed, *star bodies* are briefly studied at the end of section 3; a classical example of such a body is the domain bounded between the branches of a hyperbola. The basic problem here is to find those lattices every point of which lies outside of the star body; such a lattice is said to be *admissible.*

Next, in section 4, we introduce the notion of relative lattice movement. Using the norm defined in section 2, it becomes possible to define the concept of "almost periodicity" (in time) of such lattice motions. To take account of the mutual impenetrability of the solid particles comprising the suspension, we define *compatible* motions. This compatibility condition is shown to be equivalent to the admissibility of the initial lattice with respect to a star body deduced from the prescribed properties of both the particle and the macroscopic flow. Our analysis is necessarily limited to a system composed of identical particles.

Finally, in section 5, this star body is delineated for the whole class of isochoric two-dimensional, macroscopically homogeneous, linear (shearing) motions. We are able in such circumstances to derive a maximum value for the particle concentration as a function of a certain flow parameter, λ. Elliptic motions reproduce the periodic particle arrangement, whereas hyperbolic ones do not. The "degenerate" case of a simple shear flow, which cannot simply be deduced by limiting continuity arguments from the other two types of flow lying on either side of it, is especially interesting. In three dimensions, two types of self-reproducing configurations are possible, though they yield the same maximum concentration; in general, the motion is almost periodic, since it can be viewed as a rectilinear trajectory in a two-dimensional lattice.

2. LATTICE

2.1. *Basic notions*

Consider a suspension composed of an ordered repetitive three-dimensional array of identical rigid particles, immersed in a fluid continuum and extending indefinitely in every direction. From a formal point of view, the lattice Λ representing the group of translational self-coincidence symmetry operations of this spatially periodic medium, consists of the set of points

$$\mathbf{R}_n = n_1\mathbf{l}_1 + n_2\mathbf{l}_2 + n_3\mathbf{l}_3, \qquad [2.1]$$

where $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ are three linearly independent vectors of $\mathbb{R}^3$, serving as a basis of Λ, and $\{n_1, n_2, n_3\} \equiv \mathbf{n}$, say, are a trio of integers. The symbol $\mathbf{0} \equiv \{0, 0, 0\}$ will be chosen to designate the lattice origin.

Consider the second-order tensor

$$\mathsf{L} \overset{\text{def}}{=} \mathbf{l}_1\mathbf{e}_1 + \mathbf{l}_2\mathbf{e}_2 + \mathbf{l}_3\mathbf{e}_3, \qquad [2.2]$$

with unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denoting an orthonormal basis of the space $\mathbb{R}^2$. Define l_{ij} to be the projection $\mathbf{l}_i \cdot \mathbf{e}_j$ of the basis vector $\mathbf{l}_i$ upon the direction $\mathbf{e}_j$. Since $\mathbf{l}_i = \mathsf{L} \cdot \mathbf{e}_i \equiv \mathbf{e}_j l_{ij}$ (summation convention), eqn [2.2] may be expressed in the nonian form $\mathsf{L} = \mathbf{e}_j\mathbf{e}_i l_{ij}$. Equivalently, l_{ij} may be regarded as the element lying in the ith row and jth column of the 3×3 L matrix. The determinant $d(\Lambda)$ of the lattice Λ is

$$d(\Lambda) = |\det \mathsf{L}|; \qquad [2.3]$$

it is assumed nonzero since the triad of noncoplanar basis vectors $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ are linearly independent.

When L is proportional to the unit tensor I, namely $\mathsf{L} = l\mathsf{I}$ (with l a characteristic length), we obtain the orthonormal lattice of points with coordinates $\{n_1, n_2, n_3\}\, l$. This lattice, which plays a fundamental role in the subsequent theory, will always be designated by the symbol Y. In the general case, Λ is related to Y by the nonsingular linear transformation

$$\Lambda = LY. \qquad [2.4]$$

This may be employed to generate the most general lattice Λ by subjecting Y to such a transformation.

The preceding definitions require further commentary, which is offered below.

The formal representation of a basis by a single mathematical object is especially useful when it is desired to compare two bases. This point is elaborated upon in section 2.2. It should also be recalled that the determinant of the lattice is equal to the superficial volume τ_o of the unit cell. Thus,

$$d(\Lambda) = |\mathbf{l}_1 \times \mathbf{l}_2 \cdot \mathbf{l}_3| \equiv \tau_o. \qquad [2.5]$$

It is equally well known that L can be chosen in an infinite variety of equivalent ways. It can be shown (Lekkerkerker 1969) that a tensor L' is a basic tensor of the lattice Λ if, and only if,

$$\mathsf{L}' = \mathsf{N} \cdot \mathsf{L}, \qquad [2.6a]$$

with

$$\det \mathsf{N} = \pm 1, \qquad [2.6b]$$

wherein N is a unimodular matrix composed of integral elements. Of course, the relation [2.6] is an equivalence.

2.2. *Comparison between lattices*

The norm of a second-order tensor T may be defined by the relation

$$\|\mathsf{T}\| = \max_{i,j} |T_{ij}|. \qquad [2.7]$$

It is readily confirmed that this definition is trivially a norm in a formal sense. With the aid of this definition, the neighborhood of a lattice can be defined by the following scheme

(Lekkerkerker 1969): Let Λ be a lattice, $\mathbf{L}$ be a basis of Λ and let ϵ be a positive number. The $(\mathbf{L}, \epsilon)$ neighborhood of the lattice Λ is the set of lattices Λ' possessing a basis $\mathbf{L}'$ such that

$$\|\mathbf{L}' - \mathbf{L}\| < \epsilon. \qquad [2.8]$$

Obviously, when ϵ is very small, the lattices Λ and Λ' differ but little.

It will subsequently prove necessary for us to be able to compare two lattices, especially when they are almost identical. For example, section 5 is concerned with lattices that are "almost periodic" in time. This statement requires precise quantification. Of course, functions of $\mathbf{L}$ continuous in the norm [2.7] do not differ too much for the case of two almost identical lattices.

From a technical point of view it is, of course, important to confer upon the set of lattices the structure of a topological space; however, we shall not embark on such formal details here.

2.3. *Dirichlet, Kronecker and Weyl theorems*

This subsection addresses a particular question which seems at first rather foreign to the subject at hand, but which is, in fact, crucial to a precise understanding of the kinematics and dynamics of a periodic suspension undergoing a simple shear. Loosely stated: "What is the interaction of a straight line with the lattice Y?" As a rigorous analysis is provided elsewhere (Brenner & Adler 1985), we shall confine ourselves to a few observations, illustrated by reference to the relatively simple case of a two-dimensional lattice.

Consider the straight line

$$y = ax, \qquad [2.9]$$

of slope a. If it is assumed that a is rational, this slope can be expressed as the irreducible integer ratio p/q. The straight line [2.9] will then periodically intersect those lattice points whose coordinates are integral multiples of (q, p). This fact is illustrated in figure 1(a). The periodic character of the interaction can be emphasized even more dramatically by representing the straight line on the unit square $[0, 1]^2$, i.e. y (mod 1) as a function of x (mod 1), respectively designated by (y) and (x). This function is represented by a finite number of line segments, as depicted in figure 1(b).

The remainder of this subsection addresses three basic questions revealed by figure 1(b): (i) When a is irrational, what is the interaction of the straight line with respect to the lattice points? (ii) ... with respect to any point of the unit square? (iii) What is the pattern corresponding to figure 1(b)? These problems are solved by invoking three theorems, bearing the names of Dirichlet, Kronecker and Weyl (Hardy & Wright 1979).

When a is irrational and ϵ a positive number, a lattice point other than $\mathbf{0}$ can be found whose distance from the straight line $y = ax$ is less than ϵ. This statement can be markedly improved in accuracy by considering the trace made by the straight line $y = ax$ on the unit

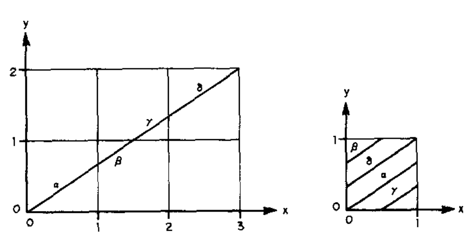

Figure 1. Interaction of a straight line with a two-dimensional lattice Y when the slope is rational. (a) In this example $p = 2, q = 3$. (b) y (mod 1) as a function of x (mod 1). The straight line passes continuously into itself.

square. This straight line is dense in $[0, 1]^2$. Whatever point of the unit square may be chosen, a trace of this line can be found arbitrarily near to the specified point. (This is obviously not true when a is rational, as is clear from figure 1(b).) In terms of visual imagery, the final unit square will appear to be grey after passage of a sufficiently large number of traces. But, and this represents the final improvement, the unit square will be *uniformly* grey, i.e. the lines are uniformly distributed on $[0, 1]^2$.

Of course, this uniform pattern is the same for any irrational value of the slope a; nor does its validity depend either upon a possible shift of origin.

Another enlightening context for viewing the problem is as a dynamic process (Arnold & Avez 1968, Parry 1981). In this sense, Weyl's theorem may be viewed as the first ergodic theorem to be discovered. Thus, consider the sequence $\{x_n\}$ of discrete points,

$$x_n = n\xi,$$

where ξ is irrational and n an integer. Then, for all continuous functions f defined on the unit interval $[0, 1]$, and satisfying the condition $f(0) = f(1)$, we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_0^1 f(y)\, dy \qquad [2.10]$$

since $\{x_n\}$ is uniformly distributed (mod 1). In other words, the dynamic process $x_n = n\xi$ is ergodic when ξ is irrational.

Finally, as is evident from the transformation [2.4], the previous results can readily be extended to the case of a general lattice Λ through the use of the affine transformation L. The easiest way to proceed is by applying the inverse transformation L^{-1} and subsequently considering the rational or irrational character of the problem.

3. BODIES ON SIMPLE LATTICES

This section is devoted to the relation between convex and star bodies and lattices. In a sense, the prior study, albeit brief, of the behavior of a straight line in an array furnished the first and simplest example of the possible relationships existing between a geometric object and a lattice.

The goal of this section is twofold: (i) to provide a quick overview of the static properties of a granular porous medium composed of identical convex particles—a situation with a broad field of applications; (ii) to provide an entrée into the kinematics of a granular periodic suspension dispersed in a uniform shear flow; for, as will be shown in sections 4 and 5, this kinematical study is closely related to the admissibility of very particular convex bodies (such as ellipsoids), or star bodies (such as hyperbolae), with respect to the lattice at a given time.

The following development is derived largely from Lekkerkerker (1969).

3.1. *Convex bodies*

A set H of $\mathbb{R}^n$ is called convex if, for every pair of points $\mathbf{x}$ and $\mathbf{y}$ belonging to H, it contains all the points of the line segment joining $\mathbf{x}$ to $\mathbf{y}$. Examples of convex bodies in $\mathbb{R}^n$ are spheres, cubes, rods and, more generally, ellipsoids and parallelotopes. As such, many, if not most, particles of physical interest are included within the class. A red blood cell, however, because of its biconcave shape, furnishes a simple example of a nonconvex body.

In view of the proposed applications to suspensions, attention will be focused primarily on spheres. Nevertheless, most of the following analysis retains its validity for general convex bodies, provided only that they possess a center of symmetry, hereafter designated as O. In most cases the lattice origin $\mathbf{0}$ is confounded with this center of symmetry. The body λK, with λ a scalar, is the set of points $\lambda\mathbf{x}$—where $\mathbf{x}$ belongs to K.

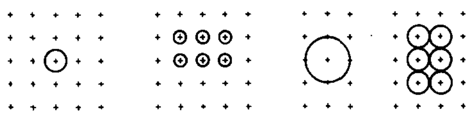

Figure 2. Admissibility and packing. In (a) the unit square lattice Y is admissible for those circles whose radius is smaller than unity; (b) shows a packing of circles of radius 1/3 along Y. In (c) the admissibility of a circle of unit radius is equivalent to the packing of circles of radius 1/2.

The central pair of concepts arising in the subsequent analysis, and which are related by an equivalence, are (i) the admissibility of a lattice for a given body, (ii) the packing of bodies according to a given lattice. A lattice Λ is called *admissible* for an arbitrary body H if it has no point other than its origin $\mathbf{0}$ contained within the interior of H. It is called *strictly admissible* if it contains no point, including $\mathbf{0}$, of H. From this, the critical determinant of the body H can be defined as

$$\Delta(H) = \inf\{d(\Lambda):\Lambda \text{ strictly admissible for } H\}. \qquad [3.1]$$

These concepts are illustrated for the case of circles in figure 2.

For K a fixed bounded o-symmetric (i.e. centrally symmetric, with symmetry center O) convex body, consider the spatially periodic medium derived from K by translations belonging to the lattice Λ; this medium may be formally represented by

$$\{K + \mathbf{x}; \mathbf{x} \in \Lambda\}, \qquad [3.2]$$

where $K + \mathbf{x}$ denotes the displacement of K by the position vector $\mathbf{x}$. This collection of bodies is called a (K, Λ) packing if no pair of bodies possess an inner point in common. This important definition is illustrated in figure 2. Note that a (K, Λ) packing is not necessarily a real physical packing (Gray 1968), for which actual contact between adjacent particles is required for mechanical stability. Here the particles do not necessarily touch. A (K, Λ) packing represents only a possible geometric arrangement of the particles. The terminological conflict here between mathematical and engineering usage of the word "packing" will be resolved in favor of the former; however, the intended meaning will usually be evident from the context.

Also illustrated by figure 2 is the fact that the concept of packing is closely related to that of admissibility. Actually, the following equivalence exists:

$$\boxed{\begin{array}{l}\text{A lattice } \Lambda \text{ is a packing lattice of } K \text{ if,}\\ \text{and only if, it is admissible for } 2K.\end{array}} \qquad [3.3]$$

This equivalence partly explains the central theoretical role played by convex bodies in the subsequent theory.

Packing concentration or density Φ represents the volumetric proportion of particles in space or, equivalently, in the unit cell. It is expressed quantitatively by the formula

$$\Phi(K, \Lambda) = V(K)/d(\Lambda), \qquad [3.4]$$

where $V(K)$ is the volume of the body K.

The *maximum* density of a lattice packing of translations of K is called the density of closest lattice packing, and is denoted by $\delta(K)$. It can easily be related to the critical determinant for the body $2K$, according to the equivalence [3.3]. This yields

$$\delta(K) = V(K)/\Delta(2K). \qquad [3.5]$$

Of course, $\Phi(K, \Lambda)$ and hence $\delta(K)$ are always less than unity. A nontrivial estimate of $\delta(K)$ requires a more precise specification of the convex body under consideration. The search for the best (i.e. the least) upper bound for $\delta(K)$ constitutes a not insignificant portion of the efforts heretofore devoted to investigations of the geometry of numbers.

Basic results concerning the packing of circles K_2 in the plane and of spheres K_3 (each of unit radius) in the Euclidean space $\mathbb{R}^3$ are

$$\Delta(K_2) = \sqrt{3}/2, \quad \delta(K_2) = \pi/2\sqrt{3}, \qquad [3.6]$$

$$\Delta(K_3) = 1/\sqrt{2}, \quad \delta(K_3) = \pi/3\sqrt{2}. \qquad [3.7]$$

Historically, these estimates for the circle and sphere are due, respectively, to Lagrange (1773) and Gauss (1831).

Finally, we briefly address the important question of invariance under an affine transformation. As already observed in the previous section pertaining to the interaction of a straight line with a lattice Λ, the effect of a nonsingular linear transformation $\mathbf{L}$ is to change the numerics, but not the essential geometrical nature of the problem. It is convenient to assign a particular role to the integral lattice Y obtained from Λ by the inverse transformation $\mathbf{L}^{-1}$. For instance, it is obvious that K contains a point other than $\mathbf{0}$ of Λ if, and only if, $\mathbf{L}^{-1}K$ contains a point other than $\mathbf{0}$ of Y. As such, the definitions of admissibility and packing remain invariant under an affine transformation. It is equally easily shown that the density too remains invariant, in particular the maximum density $\delta(K)$. Explicitly,

$$\delta(\mathbf{L}K) = \delta(K). \qquad [3.8]$$

These observations eliminate the need for essentially redundant analytical efforts. For example, it is unnecessary to consider ellipsoids separately from spheres, since ellipsoids constitute mere affine transformations of spheres. Hence, the maximum density δ for ellipsoids is the same as that for spheres.

3.2. *Star bodies*

Rather remarkably, star bodies do not occur as such in the study of static packings of convex particles, but rather arise naturally only in the kinematic study of periodic suspensions of convex particles. A star body may be defined as a set of rays S (i.e. half-lines emanating from the origin) such that if the position vector $\mathbf{x}$ belongs to S, then $\lambda\mathbf{x}$ is an inner point of S for any number λ $(0 \leq \lambda < 1)$. (More precisely, closure of S should be demanded, but this technical feature will be ignored here.)

Though a convex body is a star body, the converse is not true. The most classical example of a star body is the domain

$$S_o : |xy| \leq 1 \qquad [3.9]$$

bounded by a hyperbola, as illustrated in figure 3. As hyperbolae frequently arise in applications, attention will be focused more-or-less exclusively upon them. Evaluation of the critical determinant constitutes the most important element. It can be shown (Lekkerkerker 1969) that the critical determinant of the body S_o is

$$\Delta(S_o) = \sqrt{5}. \qquad [3.10]$$

Moreover, an example of such a critical lattice is the one generated by the two points $(1, 1)$ and $(1/2 - \sqrt{5}/2, 1/2 + \sqrt{5}/2)$, as indicated in figure 3.

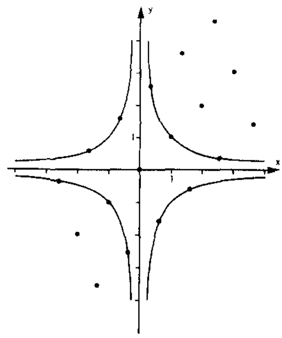

Figure 3. The domain S_o bounded by a hyperbola and a possible critical lattice.

Finally, consider the asymmetric two-dimensional domain $S_{\alpha,\beta}$, defined as

$$-\alpha \leq xy \leq \beta,$$

where α and β satisfy $\beta \geq \alpha > 0$. Of course, a first rough approximation of the critical determinant is

$$\Delta(S_{\alpha,\beta}) \geq \Delta(S_{\beta,\beta}). \qquad [3.11]$$

However, considerable improvement is possible (Lekkerkerker 1969), as revealed by the following theorem: Let α and β be two real numbers satisfying the preceding inequality. Set $\tau = \beta/\alpha$, and designate by s the largest positive integer such that $s \leq \tau$, and by t the smallest positive integer such that $t \geq \tau$. Then,

$$\Delta(S_{\alpha,\beta}) \geq \min\,[\alpha(t^2 + 4\tau)^{1/2}, \quad \alpha(\tau^2 + 4\tau^2/s)^{1/2}]. \qquad [3.12]$$

4. LATTICE MOTION AND ITS GENERAL PROPERTIES

Thus far the lattice has been considered as motionless. Henceforth, it will be regarded as deformed by a general motion, with a view towards eventually establishing the suspension's rheological properties. Lattice deformations have already been extensively studied in connection with dynamical theories of crystal lattices (Born & Huang 1954). However, kinematics will be studied in this paper without regard to its dynamical origins.

Introduction of time as an additional independent parametric variable carries with it two new notions. The first pertains exclusively to the lattice itself, and entails the concept of "almost periodicity." The second is engendered by the finite size of the particles, and involves the concept of "compatibility."

4.1. *Homogeneous isochoric motions*

A lattice is said to be in motion if the basic vectors $\mathbf{l}_i$, or equivalently the tensor $\mathbf{L}$, depend upon the time t. If a lattice is deformed in such a way that the resulting structure remains a perfect lattice, the deformation is called homogeneous. Attention will be restricted to this class of deformations. In such circumstances $\mathbf{L}$ depends only upon time, but not position. This class of lattice deformations can be characterized by the deformation gradient dyadic $\mathbf{F}$ as

$$\mathbf{l}_i(\tau) = \mathbf{F}^\dagger \cdot \mathbf{l}_i(t). \qquad [4.1a]$$

Equivalently,

$$\mathbf{L}(\tau) = \mathbf{F}^\dagger \cdot \mathbf{L}(t). \qquad [4.1b]$$

In the present rheological context, the lattice deformation may be regarded as arising from its suspension in a macroscopically homogeneous linear shear flow. The *local* fluid velocity vector field $\mathbf{v}$ at a general point $\mathbf{R}$ of such a spatially periodic suspension will then be assumed to be of the form

$$\mathbf{v}(\mathbf{R} + \mathbf{R}_n) - \mathbf{v}(\mathbf{R}) = \mathbf{R}_n \cdot \mathbf{G}, \tag{4.2}$$

where $\mathbf{G}$ is the constant macroscopic velocity gradient dyadic. In other words, the gradient $\nabla\mathbf{v}$ of the *local* velocity field is instantaneously spatially periodic for all time. For the applications we have in mind, $\mathbf{G}$ will be supposed both time independent and traceless, the latter condition requiring that

$$\operatorname{tr} \mathbf{G} = 0. \tag{4.3}$$

Consequently, the macroscopic motion is isochoric (cf. [4.6]).

Equation [4.2] may be employed to establish the time variation of $\mathbf{L}$ as

$$d\mathbf{L}/dt = \mathbf{G}^\dagger \cdot \mathbf{L}. \tag{4.4}$$

Upon integration this yields

$$\mathbf{L}(t) = (\exp \mathbf{G}^\dagger t) \cdot \mathbf{L}(0), \tag{4.5}$$

from which the deformation gradient $\mathbf{F}$ is readily deduced by comparison with [4.1b]. [Note that $\exp \mathbf{G}^\dagger t = (\exp \mathbf{G}t)^\dagger$.]

The volume of a unit cell remains constant when subjected to an isochoric motion, for according to [2.5],

$$\begin{aligned}\tau_o(t) &= \det[\mathbf{L}(t)] = [\det(\exp \mathbf{G}^\dagger t)]\tau_o(t = 0) \\ &\equiv \tau_o(t = 0),\end{aligned} \tag{4.6}$$

since $\mathbf{G}$ is traceless. As the unit cell always contains the same suspended particle volume, it is easily shown that the solids concentration ϕ in the suspension remains constant in time when it is subjected to such an isochoric motion.

4.2. *"Almost periodic" functions*

The theory of *almost periodic* functions, pioneered by H. Bohr, is essentially a generalization of the theory of periodic functions, which leaves intact the property of completeness of the corresponding Fourier series. A formal definition is as follows:

$$\boxed{\begin{array}{l}\text{A continuous function } f(t) \text{ is almost periodic if} \\ \forall\epsilon,\ \exists\tau(\epsilon),\ T \in [0, \tau(\epsilon)] / |f(t + T) - f(t)| \leq \epsilon.\end{array}} \tag{4.7}$$

Instead of developing the major consequences of this definition, which may be found, for example, in Bohr's (1947) treatise, we shall insist upon the fact that the example of section 2 (i.e. the straight line in a two-dimensional array) furnishes an example of an almost periodic function, if it is properly considered as a vectorial function of time (i.e. as a trajectory).

The equation [2.9] of a straight line can be written equivalently as the parametric curve

$$x = t, \quad y = \xi t, \tag{4.8}$$

with ξ a real coefficient. Note that $(y) = f(x)$ is a periodic function of x, though we shall not be interested in this explicit functional dependence. Rather, we shall again consider the trace,

$$(x) = (t), \qquad (y) = (\xi t), \tag{4.9}$$

of [4.8] on the unit square. Here, as in the paragraph containing [2.9], (x) and (y) respectively designate $x(\text{mod } 1)$ and $y(\text{mod } 1)$. It can be proved that the vectorial function $((x), (y))$ is an almost periodic function of time. This proof can be effected directly as an immediate consequence of Dirichlet's theorem (cf. Brenner & Adler 1985). Of course, when ξ is rational the function is periodic, since the trajectory passes over itself repeatedly [cf. figure 1(b)].

4.3. *Kinematic properties of lattices*

The two new kinematical concepts discussed below furnish the key to the eventual calculation of the suspension's rheological properties.

4.3.1. *Compatible packing*

Since the rigid suspended particles are incapable of mutual interpenetration, the lattice packing must remain a lattice packing for all time in order that the geometric existence of the configuration actually be possible at any time. This leads to the following notion of compatible packing:

A packing is compatible if it is a packing at any time for the particles.	[4.10]

Two clarifying remarks are necessitated by this definition. Consider first the case of spherical particles of radius a. The distance between two centers is necessarily larger than $4a^2$:

$$\forall \{n\}, t \qquad \mathbf{R}_n^2(t) \geq 4a^2. \tag{4.11a}$$

Equivalently, since

$$\mathbf{R}_n(t) = (\exp \mathbf{G}^\dagger t) \cdot \mathbf{R}_n(0) \equiv \mathbf{R}_n(0) \cdot \exp \mathbf{G}t,$$

this requires that

$$\mathbf{R}_n(0) \cdot (\exp \mathbf{G}t) \cdot (\exp \mathbf{G}^\dagger t) \cdot \mathbf{R}_n(0) \geq 4a^2. \tag{4.11b}$$

This inequality, in which the Cauchy-Green tensor appears, obviously depends upon two factors—namely the particular nature of the deformation gradient $\mathbf{F}$ and the initial value of $\mathbf{R}_n$.

For nonspherical particles, the compatibility condition also depends upon particle orientation. Of course, this orientation cannot be assessed by a purely kinematical study, but rather devolves upon supplemental dynamical considerations. However, two pertinent radii can be introduced to partially characterize these particles. These, ρ_{min} and ρ_{max}, are respectively defined as the corresponding radii of the spheres contained in, and containing, the particle. It is easily recognized that if a packing is compatible for the sphere ρ_{max}, it will necessarily be compatible for *any* motion of the particle. Conversely, if the packing is incompatible for the sphere ρ_{min}, it will always be incompatible for *any* motion of the particle. Though useful in their generality, these considerations may sometimes yield rather loose bounds.

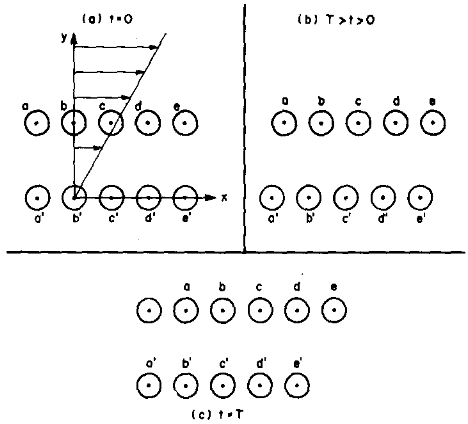

Figure 4. Periodic character of a lattice motion in a simple shear flow. The mean flow is parallel to the x axis. The lattice obviously returns to its original configuration after the elapse of a time period of duration T, though different particles are needed to effect identify of the two lattices. This corresponds to the introduction of $\mathbf{N}$ in equation [4.12].

4.3.2. *"Almost periodicity" (in time) of lattice configurations*

The second important property of deforming lattices is their ability (or inability) to reproduce themselves in time. However, exact reproducibility is too restrictive a criterion. By means of the topology introduced in section 2.2, almost periodic lattices can be defined as follows:

> A lattice motion is almost periodic if
> $\forall\, \epsilon > 0,\ \exists\ \tau(\epsilon),\ T \in [0,\tau(\epsilon)],\ \mathbf{N}(\epsilon)$ a unimodular integer matrix $/ \|\mathbf{N} \cdot \mathbf{L}(t+\tau) - \mathbf{L}(t)\| \leq \epsilon.$

[4.12]

This definition does not require that the basis of the lattice be almost periodic, but rather only that the lattice itself be almost periodic; that is to say, it is required that two bases of the lattice at different times be equivalent. This explains the introduction of the unimodular matrix $\mathbf{N}$ in [2.6]. This point is exemplified in figure 4.

Recognition has yet to be given to particle orientation in delineating the property of reproducibility. Though obviously falling outside the scope of pure kinematics, it nevertheless deserves a few comments. Suppose, for example, that the particles are ellipsoids of revolution, whose angular orientation is represented by the unit vector $\mathbf{e}$ (Brenner 1981), locked into the spheroid and lying along its symmetry axis. As a special case of [2.2], the geometry of the suspension can then be fully characterized by the four vectors $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ and $\mathbf{e}$, each of which depends upon time. One can introduce a norm on this set and subsequently define, in a fashion paralleling [4.12], the concept of almost reproducibility of the suspension. Though representing an obvious generalization, the different natures of the vectors $\mathbf{l}_i$ and $\mathbf{e}$ must be emphasized; $\mathbf{l}_i$ is known at any time from [4.1a], whereas $\mathbf{e}$ remains part of the eventual dynamical problem to be solved.

4.4. *General compatibility condition for a suspension of spherical particles*

A criterion can be derived from the definition [4.10] of a compatible lattice that can be conveniently employed for the case of spherical particles. (These spheres, of radius a, are here denoted by K.) Towards this end, introduce a complementary definition:

> A lattice is compatible if it is admissible at any time for the body $2K$.

[4.13]

The static equivalence [3.3] is obviously replaced here by the following dynamic equivalence:

A lattice Λ is compatible for the spheres K if, and only if, the (K,Λ) packing is compatible.	[4.14]

Proof of this lemma is obvious since the equivalence [3.3] is valid at any time.

Let us now demonstrate the following property:

A lattice is compatible if, and only if, it is admissible for the star body $\mathscr{S}$; that is for $\mathscr{S} = \{\mathbf{x} : \mathbf{x} = (\exp \mathbf{G}^{\dagger} t) \cdot \mathbf{x}_0,\ \forall\, \mathbf{x}_0 \in 2K,\ \forall\, t \in (-\infty, +\infty)\}$.	[4.15]

Since $2K$ is convex, and is thus a star body, it is obvious that $\mathscr{S}$ too is a star body. The sphere $2K$ is contained within this star body. Explicitly stated,

$$2K \subseteq \mathscr{S}. \qquad [4.16]$$

A distance function (Lekkerkerker 1969) can be associated to $\mathscr{S}$. In this connection, note that two points lying on the same streamline maintain the same separation distance with the origin $\mathbf{0}$; that is, this distance is time independent. Hence, if the lattice is admissible for $\mathscr{S}$, all the lattice points except $\mathbf{0}$ lie at distances larger than unity. Thus (and conversely), all the trajectories of the lattice points lie outside the sphere $2K$.

This proof, which is quite formal—but efficient—has the advantage of maximally eliminating the special role played by the time. One consequence is that the concept of compatibility is completely independent of the specific time at which the lattice compatibility is verified. Thus, compatibility can be confirmed at time t, or at any other convenient time, e.g. at time $t = t_0$, since at t_0 the lattice configuration may possess features in common with related lattices already studied in other theoretical contexts.

A useful physical interpretation of this state of affairs can be achieved by replacing the solid sphere $2K$ centered at the origin by a liquid dye-filled spherical fluid envelope. When this liquid sphere is subjected (for both negative and positive times) to the velocity gradient $\mathbf{G}$, the dyed region of the space represents the body $\mathscr{S}$. Of course, this body $\mathscr{S}$ cannot contain any lattice point.

A key element figuring in the above property is the particle's convexity, implicitly invoked via application of the equivalence [3.3]. Without this equivalence it would prove quite difficult to obtain any definite result. In a sense it would be necessary to replace *every* particle by dye, subsequently subjecting the suspension to shear. The resulting instantaneously evolving configuration would obviously prove quite difficult to comprehend, much less analyze.

Every specified motion and initial lattice configuration requires determination of the body $\mathscr{S}$ and subsequent analysis of the admissibility of the initial lattice for $\mathscr{S}$. Such knowledge permits calculation of the maximal particle volume fraction, a quantity of considerable experimental import. With $\Delta(\mathscr{S})$ the critical determinant of $\mathscr{S}$, this maximum concentration is expressible as (cf. [3.5])

$$\delta(K, \mathbf{G}) = V(K)/\Delta(\mathscr{S}). \qquad [4.17]$$

Observe (and several examples of this property will appear in section 5) that this concentration depends upon the particular bulk motion $\mathbf{G}$ undergone by the suspension.

Nevertheless, a general estimate can be obtained for $\delta(K, \mathbf{G})$. As $\mathscr{S}$ contains the sphere $2K$, the critical determinant for $\mathscr{S}$ is necessarily larger than that for $2K$. Consequently,

$$\Delta(\mathscr{S}) \gtrsim \Delta(2K). \qquad [4.18]$$

The preceding equation implies, not surprisingly, that the kinematic maximal concentration is always less than the static maximal concentration. Explicitly,

$$\delta(K, \mathbf{G}) \lesssim \delta(K, 0). \qquad [4.19]$$

This inequality, whose origin is quite obvious in the present framework, appears to have been overlooked in the past. Many semiempirical rheological constitutive equations (e.g. Mooney 1951) contain an experimentally adjustable maximum solids concentration parameter, at which concentration the viscosity effectively becomes infinite. However, the kinematic origin of this parameter appears not to have been clearly pointed out before.

5. TWO-DIMENSIONAL MOTIONS

Concepts previously introduced in the abstract will be applied in this section to the broad class of two-dimensional shear flows. These motions possess two attractive features. The first arises from their ease of generation and control in a four-roller apparatus (Mason 1977), for example. The second is that a simple parametric decomposition is known (Kao *et al.* 1977) for these motions, reducing to only three the number of separate and distinct cases requiring elaboration.

Following a brief exposition of these two-dimensional motions, an analytic expression will be given for the compatibility condition, thereby bringing to fruition one impetus for having initiated this study. For simplicity, the case will first be treated where one plane of the lattice lies in the plane of the bulk motion. Finally, the three-dimensional case will be studied.

5.1. *General description of two-dimensional motions*

Following Kao *et al.* (1977), the most general two-dimensional incompressible linear motion can be written parametrically as

$$u = Gy, \quad v = \lambda Gx, \quad w = 0, \qquad [5.1]$$

where $-1 \leq \lambda \leq 1$. Scalar G is the shear strength. All possible flow variants are encompassed by ascribing this parametric range to λ. Streamlines correspond to the set of values

$$y^2 - \lambda x^2 = \text{const}. \qquad [5.2]$$

The case $\lambda > 0$ represents a family of hyperbolas, the asymptotes of which make an angle α with the x axis, where

$$\tan \alpha = \pm \sqrt{\lambda}. \qquad [5.3]$$

Similarly, for $\lambda < 0$ the streamlines given by eqn [5.2] are a family of ellipses with axis ratio $\pm \sqrt{-\lambda}$. When $\lambda = 0$ the flow degenerates into a simple shear flow, for which the streamlines are rectilinear. Figure 5 (Kao *et al.* 1977) summarizes the entire family of trajectories.

The deformation gradient exp $\mathbf{G}t$ is easily calculated by first noting that since $\mathbf{G} = (\mathbf{ji} + \mathbf{ij}\lambda)G$,

$$\mathbf{G}^2 = \lambda G^2 \mathbf{I}_2, \qquad [5.4]$$

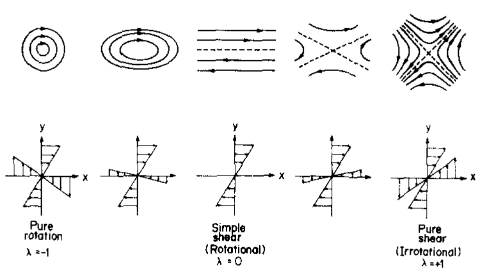

Figure 5. Schematic of a family of two-dimensional steady flows showing the streamline patterns at the top and the velocity components at the bottom. By varying λ from -1 to $+1$ the flow can be varied continuously from pure rotation (without deformation) to pure strain (with rotation).

with $\mathbf{I}_2 = \mathbf{ii} + \mathbf{jj}$ the two-dimensional idemfactor in the x-y plane. This eventually yields

$$\exp \mathbf{G}t = \mathbf{I}_2 \cosh t^* + \mathbf{G}\frac{1}{G\sqrt{\lambda}}\sinh t^* + (\mathbf{I} - \mathbf{I}_2)$$

$$\equiv \begin{pmatrix} \cosh t^* & \sqrt{\lambda}\sinh t^* & 0 \\ \frac{1}{\sqrt{\lambda}}\sinh t^* & \cosh t^* & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [5.5]$$

in which

$$t^* = \sqrt{\lambda}\, Gt. \qquad [5.6]$$

5.2. *Analytic expression of the compatibility condition*

Consider the case of spheres of radii a, whose centers are constrained to the x-y plane. According to [4.11b] the distance d between a sphere initially located at $(x = x_0, y = y_0)$ and a sphere located at the origin, is

$$d^2 = \frac{1}{2}\left\{x_0^2\left[(\cosh 2t^*)\left(1 + \frac{1}{\lambda}\right) + 1 - \frac{1}{\lambda}\right] + 2(\sinh 2t^*)\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)x_0 y_0 + y_0^2[(\cosh 2t^*)(1 - \lambda) + 1 - \lambda]\right\}. \qquad [5.7]$$

Extrema of this time-dependent function are defined by

$$\frac{d}{dt^*}(d^2) = (\sinh 2t^*)\left[x_0^2\left(1 + \frac{1}{\lambda}\right) + y_0^2(1 + \lambda)\right] + 2(\cosh 2t^*)\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)x_0 y_0 = 0, \qquad [5.8]$$

whose solutions provide the values, t_1^*, say, that extremize d. Thus, the extremal values of the distance can be expressed, for instance, as

$$d^2 \sinh 2t_1^* = \frac{1}{2}(1 - \lambda)\left(\frac{x_0^2}{\lambda} + y_0^2\right)\sinh 2t_1^* - \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)x_0 y_0, \qquad [5.9]$$

which must be larger than $4a^2$ for any value (x_0, y_0) belonging to the initial lattice.

These equations display two rather disagreeable features: (i) Calculation of the values of time at which the extrema occur requires solution of a transcendental equation, namely [5.8]; (ii) confirmation of attainment of the minimum value must be performed for an infinity of pairs of initial values, as shown by [5.9].

As a matter of chronology, it may be of interest to point out that equations [5.8] and [5.9], in fact, originally furnished the starting point for the present study—as they were quite straightforwardly written down at an early stage of the present analysis. However, we were unable to deduce any relevant information from them, except to notice the existence of the extrema cited. The novel feature of these equations was the presence of integers. This led naturally to a literature survey of the theory of numbers. In turn, this quickly led to the discovery of the existence of a whole mature field, the geometry of numbers, which then proved vital to the resolution and interpretation of these equations.

It is, of course, possible to revert to the formal solution of these equations without ever mentioning the existence of geometric number theory. It appears to us, however, that this geometry provides a fruitful, indeed necessary, structure to the physical context in which our analysis is embedded.

5.3. *Compatibility condition for a plane lattice. Reproducibility*

A plane lattice refers to a two-dimensional lattice whose two basic vectors, $\mathbf{l}_1$ and $\mathbf{l}_2$, belong to the x-y plane in which the motion [5.1] takes place. This is the simplest possible case imaginable. Yet, simultaneously, it also displays most of the characteristic features of the three-dimensional case. It may be regarded as the limiting case that occurs when the z direction of a three-dimensional lattice is sufficiently large to eliminate concerns regarding compatibility in this direction.

Hence, it now becomes necessary to consider the compatibility condition for the motion of circles of radius a, initially positioned along the lattice $\mathbf{l}_1$, $\mathbf{l}_2$. Each of the following three subsections discusses the results according to the specific class of streamline prevailing, namely $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

5.3.1. *Elliptic streamlines:* $\lambda < 0$

Streamlines here are the ellipses defined by [5.2]. It is a straightforward matter to demonstrate that the body $\mathscr{S}$ (cf. [4.15]) is the ellipse $\mathscr{E}(\lambda)$ [figure 6(a)]:

$$\mathscr{E}(\lambda):\quad y^2 - \lambda x^2 = 4a^2. \qquad [5.10]$$

Hence, the distance between the origin and any point of the lattice will always remain larger than two particle radii $2a$ if the initial lattice is admissible for this ellipse.

The maximal concentration of circles in the plane is obtained as the product of the maximal concentration of ellipses in the plane multiplied by the fractional area occupied by the circle of radius $2a$ contained within the ellipse [5.10] [see figure 6(a)]. The first term is given by [3.6], while the second is easily calculated as $\sqrt{-\lambda}$. Consequently, the maximum density of circles K_2 in the plane when the streamlines are the ellipses [5.10] is

$$\delta(K_2, \lambda < 0) = \frac{\pi}{2\sqrt{3}}\sqrt{-\lambda}. \qquad [5.11]$$

A remarkable feature of the preceding expression is that the maximum concentration depends upon the kinematic parameter λ, a characteristic already alluded to in section 4.4.

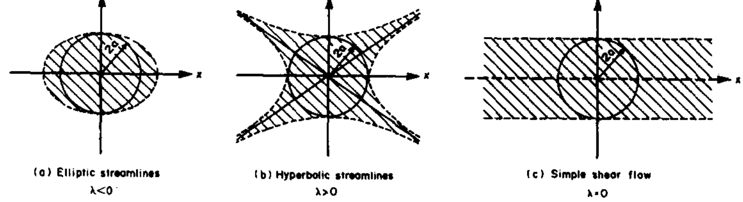

Figure 6. The star body $\mathscr{S}$ for a two-dimensional lattice subjected to a two-dimensional flow.

Note that $\delta(K_2, \lambda < 0)$ is equal to the *static* maximal concentration [3.6] for $\lambda = -1$. This fact is not surprising since it merely corresponds to a rigid body rotation of the lattice as a whole. Note further that $\delta \longrightarrow 0$ as $\lambda \longrightarrow 0$, corresponding to the case of a simple shear. This surprising feature will be further elaborated in section 5.3.2.

A "densest lattice" example is easily obtained by submitting the lattice basis $(1, 0)$, $(1/2, \sqrt{3}/2)$ to the affine transformation

$$\begin{pmatrix} \sqrt{-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \qquad [5.12]$$

which changes the circle of radius $2a$ into the ellipse $\mathscr{E}(\lambda)$.

The motion is obviously periodic (see [5.5]), possessing a period of rotation

$$T = \frac{2\pi}{G\sqrt{-\lambda}}, \qquad [5.13]$$

deriving from [5.5] and [5.6].

5.3.2. *Simple shear flow:* $\lambda = 0$

In this situation the ellipse $\mathscr{E}(\lambda)$ degenerates into the ribbon $\mathscr{R}'_o$, defined by

$$\mathscr{R}'_o: \quad |y| \leq 2a. \qquad [5.14]$$

This ribbon is an unbounded convex body. As such, according to Minkowski's theorem (Lekkerkerker 1969), it contains points of the lattice.

This degenerate case contains an ambiguity, which nicely illustrates the discontinuous character of the problem under consideration. The body $\mathscr{S}$ is not exactly the ribbon $\mathscr{R}'_o$, but rather the double ribbon

$$\mathscr{R}_o: \quad 0 < |y| \leq 2a, \qquad [5.15]$$

which arises from the fact that the fluid velocity [5.1] vanishes for $y = 0$. Thus, lattice points on the x axis become allowable, since they do not move relative to the origin.

This represents the only possibility for a compatible motion according to Dirichlet's theorem. This theorem may be used in the following manner: Employ the linear transformation $\mathbf{L}^{-1}$, which transforms the actual lattice Λ_o back into the integral lattice Y defined in section 2.1. The axis $y = 0$ is then transformed into a straight line L_o. If the slope of L_o is irrational, there then exists an infinity of lattice points lying as close as may be desired from L_o; hence, the motion is not compatible, since condition [5.15] is violated. The slope must therefore be rational, meaning that L_o may be taken as one of the basic directions of the lattice. Equivalently, when $\mathbf{L}$ is applied to Y, one axis of the lattice must necessarily lie along the x axis.

The compatibility condition may thus be enunciated as follows:

$$\boxed{\begin{array}{l} \text{A two-dimensional lattice is compatible with a} \\ \text{simple shear flow if, and only if, one of its} \\ \text{basic vectors is parallel to the flow, while} \\ \text{the absolute value of the projection of the} \\ \text{second lattice vector onto the axis} \\ \text{perpendicular to the flow exceeds } 2a. \end{array}} \qquad [5.16]$$

This condition is illustrated in figure 6(c). In an interesting sense, the simple shear greatly restricts the geometric nature of the suspension. It may be said to "organize" the suspension.

The maximum concentration is easily calculated by first packing the ribbons in the plane and then the circles within the ribbons. The areal density of ribbon packings is obviously unity, since they can be simply juxtaposed, whereas the maximum density of circles inside of a ribbon is $\pi/4$. Consequently,

$$\delta(K_2, \lambda = 0) = \pi/4. \tag{5.17}$$

Finally, as already observed in figure 4, the lattice pattern is periodic. The period is

$$T = \frac{|l_{1x}|}{G|l_{2y}|}, \tag{5.18}$$

which involves the two constant geometric characteristics of the lattice.

5.3.3. *Hyperbolic streamlines:* $\lambda > 0$

In this situation, which entails open streamlines, the star body $\mathscr{S}$ is described by the asymmetric hyperbola

$$\mathscr{H}(\lambda): \quad -4\lambda a^2 \le y^2 - \lambda x^2 \le 4a^2. \tag{5.19}$$

Application of the linear transformation $\mathbf{A}$ to the (x, y) coordinate system transforms the latter back into the standard case,

$$\mathbf{A}: \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2a}\begin{pmatrix} \sqrt{\lambda} & 1 \\ -\sqrt{\lambda} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.20}$$

of section 3.2, whose determinant is

$$\det(\mathbf{A}) = \frac{\sqrt{\lambda}}{2a^2}. \tag{5.21}$$

In this new coordinate system the star body $\mathscr{H}(\lambda)$ is expressed as

$$-\lambda \le XY \le 1. \tag{5.22}$$

In the present case λ belongs to the interval [0, 1], whence an integer n can always be found such that

$$\frac{1}{n+1} \le \lambda \le \frac{1}{n}. \tag{5.23}$$

Hence, an immediate application of [3.12] yields the following estimate for the critical determinant:

$$\Delta(\lambda) \ge \min[m_1(\lambda), m_2(\lambda)], \tag{5.24}$$

wherein

$$m_1(\lambda) = \lambda\left[(n+1)^2 + \frac{4}{\lambda}\right]^{1/2} \tag{5.25a}$$

and

$$m_2(\lambda) = \left(1 + \frac{4}{n}\right)^{1/2}. \qquad [5.25b]$$

It can readily be shown that

$$m_1\left(\frac{1}{n+1}\right) < m_2\left(\frac{1}{n+1}\right) \qquad [5.26a]$$

and

$$m_1\left(\frac{1}{n}\right) > m_2\left(\frac{1}{n}\right), \qquad [5.26b]$$

provided that we remain within the interval [5.23]. Hence, in every such interval, both possibilities [5.25] need be entertained. They become equal somewhere in the middle of the interval [5.23], namely at the value

$$\lambda(n) = \frac{1}{(n+1)^2}\left\{\left[4 + (n+1)^2\left(1 + \frac{4}{n}\right)\right]^{1/2} - 2\right\} \quad (n = 1, \ldots, \infty). \qquad [5.27]$$

Estimates m_1 and m_2 for two successive intervals [5.23] coincide at the value of λ common to these two intervals. Hence, the function $\Delta(\lambda)$ is continuous.

The maximum concentration is easily estimated from the critical determinant to be

$$\delta(\lambda) \leq \frac{\pi\sqrt{\lambda}}{2\Delta(\lambda)}. \qquad [5.28]$$

More precisely, the following two possibilities exist:

$$\lambda \in \left[\frac{1}{n+1}, \lambda(n)\right], \quad \delta(\lambda) \leq \frac{\pi}{2[\lambda(n+1)^2 + 4]^{1/2}}; \qquad [5.29a]$$

$$\lambda \in \left[\lambda(n), \frac{1}{n}\right], \quad \delta(\lambda) \leq \frac{\pi\sqrt{\lambda}}{2\left(1 + \frac{4}{n}\right)^{1/2}}. \qquad [5.29b]$$

This upper limit possesses several interesting features: It tends to zero when λ does; again, a discontinuity occurs for the limiting simple shear case, and for the same reason as previously stated. This function, which is portrayed in figure 7, is not always differentiable. It has the appearance of a saw, whose teeth move closer together as $\lambda \rightarrow 0$. Simultaneously,

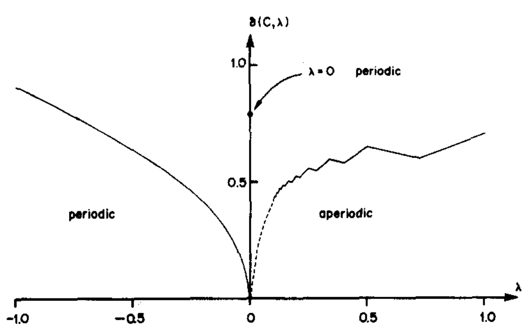

Figure 7. The maximum density $\delta(K_2, \lambda)$ as a function of the flow parameter λ for a two-dimensional lattice.

the teeth become smaller. Such pathological behavior is rarely encountered in physical phenomena.

Simpler estimates, albeit looser, can be obtained for $\delta(\lambda)$. For instance, estimate [5.29b] may be employed for the whole interval [0, 1]. This corresponds, in fact, to an estimate by Segré of the critical determinant, as reported in Lekkerkerker (1969).

Finally, it is evident that the lattice cannot be self-reproducing in such hyperbolic flows. In the XY coordinate system, defined by [5.20], the deformation gradient is diagonalized, whence the lattice points are transformed as

$$X = e^{t^*}X_0, \quad Y = e^{-t^*}Y_0. \qquad [5.30]$$

Obviously, the reproducibility criterion [4.12] can never be satisfied.

5.3.4. *Conclusions*

We may now gather all the information obtained in section 5.3, and thereby represent the maximum concentration as a function of the kinematic parameter λ, together with the reproducibility of the flow in figure 7. As already observed, this density is a discontinuous function of λ in the neighborhood of $\lambda = 0$. This discontinuity is specially interesting since it yields two different limiting density values, according as $\lambda \to 0+$ or $\lambda \to 0-$. The asymmetric variation with λ of the density derives from the two different patterns prevailing for $\lambda > 0$ and $\lambda < 0$, corresponding respectively, to open and closed trajectories.

5.4. *Compatibility condition for a three-dimensional lattice. Reproducibility*

The approach espoused in section 5.3 is here extended to a three-dimensional lattice. This extension is straightforward, at least in principle, for elliptic and hyperbolic flows. Some new features, however, arise for the case of a simple shear flow.

5.4.1. *Elliptic streamlines:* $\lambda < 0$

In the present situation the body $\mathscr{S}$ becomes the three-dimensional ellipsoid

$$\mathscr{E}'(\lambda): \quad y^2 + z^2 - \lambda x^2 \leq 4a^2, \qquad [5.31]$$

which is represented in figure 8. In a manner strictly paralleling that used in section 5.3.1, the maximum concentration of spheres (K_3) is found to be

$$\delta(K_3, \lambda < 0) = \frac{\pi}{3\sqrt{2}}\sqrt{-\lambda}. \qquad [5.32]$$

Other features of the problem remain trivially the same as for the two-dimensional case, especially the periodic character of the lattice.

5.4.2. *Simple shear flow:* $\lambda = 0$

As for two-dimensional lattices, self-reproducing motion is not possible in general unless it becomes degenerate via a particular relative configuration of the velocity field with respect to the lattice. However, in three dimensions two possible degeneracies exist: (i) The direction of the flow is parallel to a lattice plane; (ii) the direction of the flow is parallel to a lattice line.

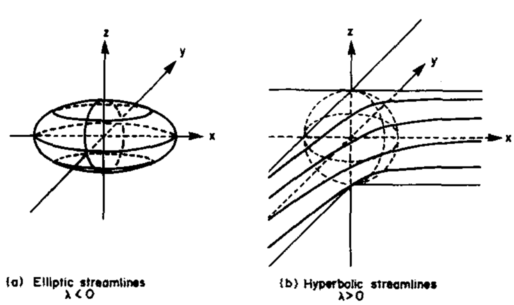

Figure 8. The star body $\mathscr{S}$ for a three-dimensional lattice subjected to a two-dimensional flow.

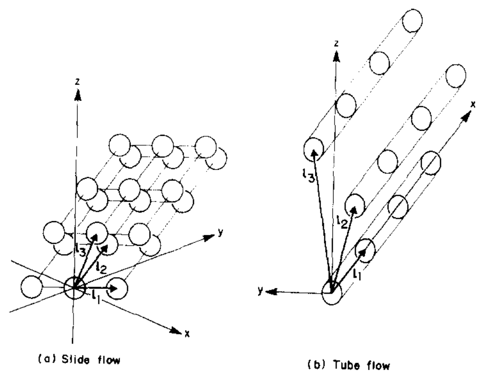

Figure 9. The two possible configurations for a three-dimensional lattice in a simple shear flow. The velocity field is given by $u = Gz$.

In the first situation the flow will be termed a *slide flow;* in the second, a *tube flow.* Reasons for selecting these names will become evident.

An investigation of the properties of these two possible flows now follows.

5.4.2.1. *Slide flow*

Suppose, as depicted in figure 9(a), that the direction of the shear flow is parallel to the x-y plane, which also contains the lattice vectors $\mathbf{l}_1$ and $\mathbf{l}_2$. Without any loss of generality we may write

$$u = Gz, \quad v = 0, \quad w = 0. \qquad [5.33]$$

Hence, no relative particle motion occurs in a plane perpendicular to z. Vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ thus remain constant in time. Accordingly, the geometry of the lattice is characterized by the position of the vector $\mathbf{l}_3$ alone. Since the projection of $\mathbf{l}_3$ upon the z axis remains constant, only its projection onto the x-y plane need be known. But the trajectory of the projection of $\mathbf{l}_3$ upon the x-y plane is a straight line. Thereby we are led back to the canonical problem studied in section 2 of the behavior of a straight line within the $(\mathbf{l}_1, \mathbf{l}_2)$ lattice.

According to Dirichlet's theorem the projection of $\mathbf{l}_3$ onto the x-y plane may be as close as desired to a specified point of the two-dimensional lattice $(\mathbf{l}_1, \mathbf{l}_2)$. In order that the motion remain compatible the projection of $\mathbf{l}_3$ onto z must be larger than $2a$. Explicitly,

$$| l_{33} | \geq 2a. \qquad [5.34]$$

A way exists of envisioning the physical realization of this property, namely the imagery of slabs sliding over one another. Hence, the choice of the name assigned to the flow. This picture permits an elementary calculation of the maximum concentration. Since the slabs can be juxtaposed, as were the ribbons in the plane in section 5.3.2, their maximal concentration is unity. On the other hand, the maximal concentration in a slab derives from a hexagonal pattern, and is thus equal to $\pi/3\sqrt{3}$. Hence (see also equation [5.35b]),

$$\delta(K_3, \lambda = 0) = \frac{\pi}{3\sqrt{3}}. \qquad [5.35a]$$

According to the almost periodicity condition [4.12], interest exists only in the projection of $\mathbf{l}_3$ upon the x-y plane, modulo $(\mathbf{l}_1, \mathbf{l}_2)$. As a direct consequence of section 4.2, such a function is known to be almost periodic. Hence, in general, the sliding motion of a lattice is almost reproducible, a property which generalizes the reproducibility of a two-dimensional lattice in a simple shear flow.

5.4.2.2. *Tube flow*

Consider the second possibility, where the flow is parallel to the lattice vector $\mathbf{l}_1$, as shown in figure 9(b). This flow is again assumed to be of the general form [5.33]. Visualization of the suspension motion can therefore be achieved by embedding it in the context of a series of tubes having axes lying perpendicular to the y-z plane, with the spheres slipping through these tubes. Note that no relative motion of sphere pairs exists within a given tube, and no lateral motion occurs in the y-z plane. The velocity of the spheres is proportional to the tube elevation z, or equivalently to the distances l_{23} and l_{33}. The imagery here is slightly more difficult to visualize than that of the previous configuration.

Via this representation the maximal sphere concentration is easily shown to be equal to the maximum concentration of cylinders in $\mathbb{R}^3$, multiplied by the maximum concentration of spheres in a cylinder of identical radius. The first concentration is achieved with a hexagonal pattern, and is equal to $\pi/2\sqrt{3}$. The second is equal to $2/3$. Therefore, in tube flow the maximum concentration is

$$\delta(K_3, \lambda = 0) = \frac{\pi}{3\sqrt{3}}. \qquad [5.35b]$$

That this value is identical to that derived for a slide flow is remarkable.

In order to be able to represent the positions of the spheres it suffices to know only the projections of the lattice vectors $\mathbf{l}_2$ and $\mathbf{l}_3$ onto the x axis. These projections,

$$\begin{aligned} l_{21} &= Gl_{23}t + \text{const}, \\ l_{32} &= Gl_{33}t + \text{const}, \end{aligned} \qquad [5.36]$$

are linear functions of time, with coefficients proportional to the vertical coordinate of the tube. This motion can be represented by a straight line within a plane. Moreover, interest exists in the configuration only modulo l_{11} in both directions. Hence, exactly the same situation prevails as before. As such, the current motion is also almost periodic.

5.4.2.3. *Conclusions*

The preceding conclusions can be coalesced into a single proposition, namely

> Self-reproducing motion of three-dimensional spherical-particle suspensions in simple shear flow is only possible in either of two configurations, termed tube flow and slide flow. In both cases the maximum concentration is
>
> $$\delta(K_3, \lambda = 0) = \frac{\pi}{3\sqrt{3}}.$$
>
> Additionally, the motion is almost periodic in both cases.

[5.37]

5.4.3. *Hyperbolic streamlines:* $\lambda > 0$

For this case the body $\mathscr{S}$ becomes the "four-legged starfish,"

$$\mathcal{H}'(\lambda): \quad -\lambda(4a^2 - z^2) \leq y^2 - \lambda x^2 \leq 4a^2 - z^2, \qquad [5.38]$$

which is represented in figure 8(b).

Apply to the (x, y, z) coordinate system the transformation

$$\mathbf{A}': \quad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \frac{1}{2a}\begin{pmatrix} \sqrt{\lambda} & 1 & 0 \\ -\sqrt{\lambda} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad [5.39]$$

representing the three-dimensional counterpart of [5.20]. This transformation possesses the determinant

$$\det(\mathbf{A}') = \frac{\sqrt{\lambda}}{4a^3}. \qquad [5.40]$$

In terms of this new coordinate system the star body $\mathcal{H}'(\lambda)$ is described by the equation

$$-\lambda(1 - Z^2) \leq XY \leq 1 - Z^2. \qquad [5.41]$$

To our knowledge no relevant study of this body exists.

A simple estimate of the critical determinant may be derived as follows. Let us seek a critical lattice two of whose vectors are parallel to the X-Y plane. Hence, these two vectors, $\mathbf{l}_1$ and $\mathbf{l}_2$, are necessarily those previously obtained in the two-dimensional case studied in section 5.3.3.

Consider the plane $Z = \sqrt{3}/2$. The section of the body $\mathcal{S}$ intersected by this plane is

$$-\lambda/4 \leq XY \leq 1/4. \qquad [5.42]$$

The centered lattice $(\mathbf{l}_1/2, \mathbf{l}_2/2)$ is admissible for this star body. Consequently, the lattice

$$\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\left(\frac{l_{11} + l_{21}}{2}, \frac{l_{12} + l_{22}}{2}, \frac{\sqrt{3}}{2}\right) \qquad [5.43]$$

is admissible for $\mathcal{H}'(\lambda)$. Note that within the planes $\pm\sqrt{3}/2$, this lattice is identical to the lattice $(\mathbf{l}_1/2, \mathbf{l}_2/2)$, in which every other point has been suppressed. The corresponding value $\Delta'(\lambda)$ of the critical determinant is then

$$\Delta'(\lambda) = \frac{\sqrt{3}}{2}\Delta(\lambda), \qquad [5.44]$$

with $\Delta(\lambda)$ given by [5.24]. (Better estimates than [5.24] almost certainly exist, but the effort expended by venturing into such a technical problem may not result in sufficient improve-

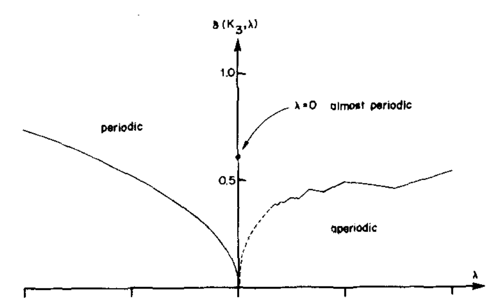

Figure 10. The maximum density $\delta(K_3, \lambda)$ as a function of the flow parameter λ for a three-dimensional lattice.

ment to prove worthwhile.) Hence, the estimate derived for the maximum density in a hyperbolic flow is

$$\delta(K_3, \lambda > 0) = \frac{4}{3\sqrt{3}}\delta(C, \lambda > 0). \qquad [5.45]$$

Lattice reproducibility does not obtain in such a flow, since by a trivial extension of [5.30] the lattice points transform as

$$X = e^{t^*}X_0, \quad Y = e^{-t^*}Y_0, \quad Z = Z^*. \qquad [5.46]$$

5.4.4. *Conclusions*

All information obtained for three-dimensional lattices is gathered together in figure 10. Most comments offered in section 5.3.4 could be repeated here with equal veracity. Some interesting new features are introduced by simple shear flows. Observe that when compared with the two-dimensional lattice, the maximum density is multiplied by about the same factor, namely, 0.77, for each of the three types of flow.

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NOMENCLATURE

a	slope of a straight line in eqn [2.9] or radius of a spherical particle
$\mathbf{A}, \mathbf{A}'$	square matrices defined in eqns [5.20] and [5.39]
d	distance between sphere centers defined in eqn [5.7]
d or det	determinant
$\mathbf{e}$	unit vector along the symmetry axis of a body of revolution
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	unit vectors forming an orthonormal basis of Euclidean space $\mathbb{R}^3$
$\mathcal{E}; \mathcal{E}'$	ellipse; ellipsoid
f	continuous function
$\mathbf{F}$	deformation gradient dyadic of a lattice defined in eqn [4.1]
G	scalar shear rate for a two-dimensional shear flow defined in eqn [5.1]
$\mathbf{G}$	macroscopic velocity gradient dyadic defined in eqn [4.2]
H	generic convex body
$\mathcal{H}, \mathcal{H}'$	two- and three-dimensional hyperbolic bodies defined in eqns [5.19] and [5.38]
$\mathbf{I}$	dyadic idemfactor
$\mathbf{I}_2$	two-dimensional idemfactor in a plane
K_2	circle
$K; K_3$	o-symmetric (centrosymmetric body with an origin O) convex body; sphere
$\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$	basic lattice vectors of the lattice Λ
l_{ij}	projection of the basic lattice vector $\mathbf{l}_i$ onto the jth direction
$L; L_o$	nonsingular affine transformation of a lattice Λ, defined by eqn [2.4]; straight line
$\mathbf{L}$	3×3 matrix, dyadic or second-order tensor representation of a three-dimensional lattice, defined in eqn [2.2]
$m_1(\lambda), m_2(\lambda)$	function of λ defined in eqn [5.25]
n, N	integers
$\{n_1, n_2, n_3\} \equiv \mathbf{n}$	triplet of integers locating a lattice point $\mathbf{R_n}$ or a unit cell $\mathbf{n}$ or $\{\mathbf{n}\}$
$\mathbf{N}$	unimodular matrix composed of integers, either positive, negative or zero
o or O	center of symmetry of a centrally-symmetric convex body

p, q	integers
$\mathbf{R}$	generic position vector of a point
$\mathbf{R}_n$	position vector of a lattice point or unit cell
$\mathcal{R}'_o$; $\mathcal{R}_o$	ribbon; double ribbon
$\mathcal{S}$; $\mathcal{S}'$	star bodies defined in section 3.2 and eqn [4.15]
$S_{\alpha,\beta}$	two-dimensional hyperbolic domain
t	time
t^*	dimensionless time defined in eqn [5.6]
tr	trace
T	period of a periodic or almost periodic function
$\mathbf{T}$, T_{ij}	second-order tensor
u, v, w	velocity components in the (x, y, z) directions, respectively
$\mathbf{v}$, $\mathbf{v}(\mathbf{R})$	local fluid velocity vector field at point $\mathbf{R}$
$V(K)$	volume of convex body K
x, y, z	Cartesian coordinates
$(x), (y)$	designates x (mod 1) and y (mod 1), as in Fig. 1(b)
x_n	discrete points
$\mathbf{x}$	position vector of a point in $\mathbb{R}^n$
X, Y, Z	Cartesian coordinate system defined by eqn [5.20] or [5.39]
Y	fundamental integral lattice Λ such that $\mathbf{L}$ defined in eqn [2.2] is the unit diagonal tensor

Greek letters

$\delta(K)$, $\delta(K,\lambda)$	maximum possible density of a lattice packing of identical convex bodies K for the static and shear (λ) cases, respectively
$\Delta(H)$, $\Delta(\lambda)$	critical determinant of a body H for the static and shear (λ) cases, defined in eqns [3.1] and [5.24], respectively
ϵ	small positive parameter
λ	parameter possessing the property that $\lvert \lambda \rvert \leq 1$, and corresponding physically to eqn [5.1] and fig. 5
Λ	general or admissible lattice
ξ	real number, rational or irrational
τ, $\tau(\epsilon)$	time
τ_o	volume of unit cell, defined in eqn [2.5]
ϕ	volume fraction of particles in space, defined in eqn [3.4]

Special symbols

$\mathbf{0}$	origin $\{0, 0, 0\}$ of a lattice
-1	inverse operator
$\dagger$	transposition operator
$\Vert\ \Vert$	norm of a dyadic, defined in eqn [2.7]

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